

A TORELLI TYPE THEOREM FOR EXP-ALGEBRAIC CURVES

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ABSTRACT. An exp-algebraic curve consists of a compact Riemann surface S together with n equivalence classes of germs of meromorphic functions modulo germs of holomorphic functions, $\mathcal{H} = \{[h_1], \dots, [h_n]\}$, with poles of orders $d_1, \dots, d_n \geq 1$ at points p_1, \dots, p_n . This data determines a space of functions $\mathcal{O}_{\mathcal{H}}$ (respectively, a space of 1-forms $\Omega_{\mathcal{H}}^0$) holomorphic on the punctured surface $S' = S - \{p_1, \dots, p_n\}$ with exponential singularities at the points p_1, \dots, p_n of types $[h_1], \dots, [h_n]$, i.e., near p_i any $f \in \mathcal{O}_{\mathcal{H}}$ is of the form $f = ge^{h_i}$ for some germ of meromorphic function g (respectively, any $\omega \in \Omega_{\mathcal{H}}^0$ is of the form $\omega = \alpha e^{h_i}$ for some germ of meromorphic 1-form).

For any $\omega \in \Omega_{\mathcal{H}}^0$ the completion of S' with respect to the flat metric $|\omega|$ gives a space $S^* = S' \cup \mathcal{R}$ obtained by adding a finite set \mathcal{R} of $\sum_i d_i$ points, and it is known that integration along curves produces a nondegenerate pairing of the relative homology $H_1(S^*, \mathcal{R}; \mathbb{C})$ with the deRham cohomology group defined by $H_{dR}^1(S, \mathcal{H}) := \Omega_{\mathcal{H}}^0 / d\mathcal{O}_{\mathcal{H}}$.

There is a degree zero line bundle $L_{\mathcal{H}}$ associated to an exp-algebraic curve, with a natural isomorphism between $\Omega_{\mathcal{H}}^0$ and the space $W_{\mathcal{H}}$ of meromorphic $L_{\mathcal{H}}$ -valued 1-forms which are holomorphic on S' , so that $H_1(S^*, \mathcal{R}; \mathbb{C})$ maps to a subspace $K_{\mathcal{H}} \subset W_{\mathcal{H}}^*$. We show that the exp-algebraic curve (S, \mathcal{H}) is determined uniquely by the pair $(L_{\mathcal{H}}, K_{\mathcal{H}} \subset W_{\mathcal{H}}^*)$.

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1. INTRODUCTION

A choice of nonconstant meromorphic function z on a compact Riemann surface S realizes S as a finite sheeted branched covering of the Riemann sphere $\hat{\mathbb{C}}$. *Log-Riemann surfaces of finite type* are certain branched coverings, in a generalized sense, of \mathbb{C} by a punctured compact Riemann surface, namely, which are given by certain transcendental functions of infinite degree. Formally a log-Riemann surface consists of a Riemann surface together with a local holomorphic diffeomorphism π from the surface to \mathbb{C} such that the set of points \mathcal{R} added to the surface, in the completion $S^* = S' \cup \mathcal{R}$ with respect to the path-metric induced by the flat metric $|d\pi|$, is discrete. Log-Riemann surfaces were defined and studied in [BPM15a] (see also [BPM06]), where it was shown that the map π restricted to any small enough

punctured metric neighbourhood of a point w^* in \mathcal{R} gives a covering of a punctured disc in \mathbb{C} , and is thus equivalent to either $(z \mapsto z^n)$ restricted to a punctured disc $\{0 < |z| < \epsilon\}$ (in which case we say w^* is a ramification point of order n) or to $(z \mapsto e^z)$ restricted to a half-plane $\{\Re z < C\}$ (in which case we say w^* is a ramification point of infinite order).

A log-Riemann surface is said to be of finite type if it has finitely many ramification points and finitely generated fundamental group. We will only consider those for which the set of infinite order ramification points is nonempty (otherwise the map π has finite degree and is given by a meromorphic function on a compact Riemann surface). In [BPM15b], [BPM13], uniformization theorems were proved for log-Riemann surfaces of finite type, which imply that a log-Riemann surface of finite type is given by a pair $(S' = S - \{p_1, \dots, p_n\}, \pi)$, where S is a compact Riemann surface, and π is a meromorphic function on the punctured surface S' such that the differential $d\pi$ has essential singularities at the punctures of a specific type, namely *exponential singularities*.

Given a germ of meromorphic function h at a point p of a Riemann surface, a function f with an isolated singularity at p is said to have an exponential singularity of type h at p if locally $f = ge^h$ for some germ of meromorphic function g at p , while a 1-form ω is said to have an exponential singularity of type h at p if locally $\omega = \alpha e^h$ for some germ of meromorphic 1-form α at p . Note that the spaces of germs of functions and 1-forms with exponential singularity of type h at p only depend on the equivalence class $[h]$ in the space $\mathcal{M}_p/\mathcal{O}_p$ of germs of meromorphic functions at p modulo germs of holomorphic functions at p .

Thus the uniformization theorems of [BPM15b], [BPM13] give us n germs of meromorphic functions h_1, \dots, h_n at the punctures p_1, \dots, p_n , with poles of orders $d_1, \dots, d_n \geq 1$ say, such that near a puncture p_j the map π is of the form $\int g_j e^{h_j} dz$, where g_j is a germ of meromorphic function near p_j and z a local coordinate near p_j . The punctures correspond to ends of the log-Riemann surface, where at each puncture p_j , d_j infinite order ramification points are added in the metric completion, so that the total number of infinite order ramification points is $\sum_j d_j$. The d_j infinite order ramification points added at a puncture p_j correspond to the d_j directions of approach to the puncture along which $\Re h_j \rightarrow -\infty$ so that e^{h_j} decays exponentially and $\int^z g_j e^{h_j} dz$ converges. In the case of genus zero with one puncture for example, which is considered in [BPM15b], π must have the form $\int R(z) e^{P(z)} dz$ where R is a rational function and P is a polynomial of degree equal to the number of infinite order ramification points.

In [Bis16], certain spaces of functions and 1-forms on a log-Riemann surface S^* of finite type were defined, giving rise to a deRham cohomology group $H_{dR}^1(S^*)$. The integrals of the 1-forms considered along curves in S^* joining the infinite ramification points converge, giving rise to a pairing between $H_{dR}^1(S^*)$ and $H_1(S^*, \mathbb{R}; \mathbb{C})$, which was shown to be nondegenerate ([Bis16]).

The spaces of functions and 1-forms defined were observed to depend only on the types h_1, \dots, h_n of the exponential singularities, and so a notion less rigid than that of a log-Riemann surface was defined, namely the notion of an *exp-algebraic curve*, which consists of a compact Riemann surface S together with n equivalence classes of germs of meromorphic functions modulo germs of holomorphic functions,

$\mathcal{H} = \{[h_1], \dots, [h_n]\}$, with poles of orders $d_1, \dots, d_n \geq 1$ at points p_1, \dots, p_n . The relevant spaces of functions and 1-forms with exponential singularities at p_1, \dots, p_n of types $[h_1], \dots, [h_n]$ can then be defined as follows:

$$\begin{aligned}\mathcal{M}_{\mathcal{H}} &:= \{f \mid f \text{ meromorphic function on } S' \text{ with exponential singularities} \\ &\quad \text{of types } [h_1], \dots, [h_n]\} \\ \mathcal{O}_{\mathcal{H}} &:= \{f \in \mathcal{M}_{\mathcal{H}} \mid f \text{ holomorphic on } S'\} \\ \Omega_{\mathcal{H}} &:= \{\omega \mid \omega \text{ meromorphic 1-form on } S' \text{ with exponential singularities} \\ &\quad \text{of types } [h_1], \dots, [h_n]\} \\ \Omega_{\mathcal{H}}^0 &:= \{\omega \in \Omega_{\mathcal{H}} \mid \omega \text{ holomorphic on } S'\}.\end{aligned}$$

For $f \in \mathcal{M}_{\mathcal{H}}$ (respectively, $\omega \in \Omega_{\mathcal{H}}$) we can define a divisor $(f) = \sum_{p \in S} n_p \cdot p$ (respectively, $(\omega) = \sum_{p \in S} m_p \cdot p$) by $n_p = \text{ord}_p(f)$ if $p \in S'$ and $n_p = \text{ord}_{p_i}(g)$ if $p = p_i$, where g is a germ of meromorphic function at p_i such that $f = ge^{h_i}$ (respectively, $m_p = \text{ord}_p(\omega)$ if $p \in S'$ and $n_p = \text{ord}_{p_i}(\alpha)$ if $p = p_i$, where α is a germ of meromorphic 1-form at p_i such that $\omega = \alpha e^{h_i}$).

In [Bis16] it is shown how to naturally associate to an exp-algebraic curve (S, \mathcal{H}) a degree zero line bundle $L_{\mathcal{H}}$ together with a meromorphic connection $\nabla_{\mathcal{H}}$ with poles at p_1, \dots, p_n . The connection 1-form of $\nabla_{\mathcal{H}}$ near p_i is given (with respect to an appropriate local trivialization) by dh_i , so that the pair $(L_{\mathcal{H}}, \nabla_{\mathcal{H}})$ determines the exp-algebraic curve (S, \mathcal{H}) . There are naturally defined isomorphisms between the space of meromorphic sections of $L_{\mathcal{H}}$ (respectively, the space of meromorphic $L_{\mathcal{H}}$ -valued 1-forms) and $\mathcal{M}_{\mathcal{H}}$ (respectively, $\Omega_{\mathcal{H}}$), such that a meromorphic section s of $L_{\mathcal{H}}$ (respectively, a meromorphic $L_{\mathcal{H}}$ -valued 1-form α) maps to an $f \in \mathcal{M}_{\mathcal{H}}$ with the same divisor as s (respectively, an $\omega \in \Omega_{\mathcal{H}}$ with the same divisor as α).

In particular the space $W_{\mathcal{H}}$ of meromorphic $L_{\mathcal{H}}$ -valued 1-forms which are holomorphic on S' is naturally isomorphic to the space $\Omega_{\mathcal{H}}^0$. Fixing an $f \in \mathcal{O}_{\mathcal{H}}$ inducing a log-Riemann surface structure on S , with completion $S^* = S' \cup \mathcal{R}$, the 1-forms in $\Omega_{\mathcal{H}}^0$ can be integrated along curves in $H_1(S^*, \mathcal{R}; \mathbb{C})$, giving a map

$$H_1(S^*, \mathcal{R}; \mathbb{C}) \rightarrow (\Omega_{\mathcal{H}}^0)^* \simeq W_{\mathcal{H}}^*.$$

Let $K_{\mathcal{H}} \subset W_{\mathcal{H}}^*$ denote the image of $H_1(S^*, \mathcal{R}; \mathbb{C})$ in $W_{\mathcal{H}}^*$. Then our Torelli-type theorem for exp-algebraic curves states that the pair $(L_{\mathcal{H}}, K_{\mathcal{H}})$ determines the exp-algebraic curve (S, \mathcal{H}) :

Theorem 1.1. *Let $(S, \mathcal{H}_1), (S, \mathcal{H}_2)$ be two exp-algebraic curves with the same underlying Riemann surface S , and the same set of punctures p_1, \dots, p_n . Suppose that $H_1(S_1^*, \mathcal{R}; \mathbb{C})$ is nontrivial, that the line bundles $L_{\mathcal{H}_1}, L_{\mathcal{H}_2}$ are isomorphic and that the induced isomorphism $W_{\mathcal{H}_1}^* \rightarrow W_{\mathcal{H}_2}^*$ maps $K_{\mathcal{H}_1}$ to $K_{\mathcal{H}_2}$. Then $\mathcal{H}_1 = \mathcal{H}_2$.*

Finally, we mention briefly some appearances of functions with exponential singularities in the literature. Certain functions with exponential singularities, namely the n -point *Baker-Akhiezer functions* ([Bak28], [Akh61]), have been used in the algebro-geometric integration of integrable systems (see, for example, [Kri76], [Kri77a] and the surveys [Kri77b], [Dub81], [KN80], [DKN85]). Given a divisor D on S' , an n -point Baker-Akhiezer function (with respect to the data $(\{p_j\}, \{h_j\}, D)$)

is a function f in the space $\mathcal{M}(S^*)$ satisfying the additional properties that the divisor (f) of zeroes and poles of f on S' satisfies $(f) + D \geq 0$, and that $f \cdot e^{-h_j}$ is holomorphic at p_j for all j . For D a non-special divisor of degree at least g , the space of such Baker-Akhiezer functions is known to have dimension $\deg D - g + 1$.

Functions and differentials with exponential singularities on compact Riemann surfaces have also been studied by Cutillas Ripoll ([Cut84], [Cut89], [Cut90]), where they arise naturally in the solution of the *Weierstrass problem* of realizing arbitrary divisors on compact Riemann surfaces, and by Taniguchi ([Tan01], [Tan02]), where entire functions satisfying certain topological conditions (called “structural finiteness”) are shown to be precisely those entire functions whose derivatives have an exponential singularity at ∞ , namely functions of the form $\int Q(z)e^{P(z)} dz$, where P, Q are polynomials.

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2. LOG-RIEMANN SURFACES OF FINITE TYPE AND EXP-ALGEBRAIC CURVES

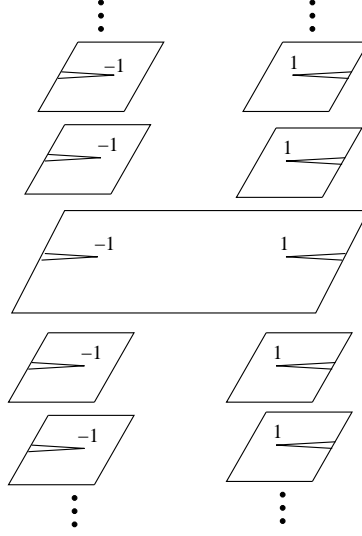
We recall some basic definitions and facts from [BPM15a], [BPM15b], [BPM13].

Definition 2.1. A log-Riemann surface is a pair (S, π) , where S is a Riemann surface and $\pi : S \rightarrow \mathbb{C}$ is a local holomorphic diffeomorphism such that the set of points \mathcal{R} added to S in the completion $S^* := S \sqcup \mathcal{R}$ with respect to the path metric induced by the flat metric $|d\pi|$ is discrete.

The map π extends to the metric completion S^* as a 1-Lipschitz map. In [BPM15a] it is shown that the map π restricted to a sufficiently small punctured metric neighbourhood $B(w^*, r) - \{w^*\}$ of a ramification point is a covering of a punctured disc $B(\pi(w^*), r) - \{\pi(w^*)\}$ in \mathbb{C} , and so has a well-defined degree $1 \leq n \leq +\infty$, called the order of the ramification point (we assume that the order is always at least 2, since order one points can always be added to S and π extended to these points in order to obtain a log-Riemann surface).

Definition 2.2. A log-Riemann surface is of finite type if it has finitely many ramification points and finitely generated fundamental group.

For example, the log-Riemann surface given by $(\mathbb{C}, \pi = e^z)$ is of finite type (with the metric $|d\pi|$ it is isometric to the Riemann surface of the logarithm, which is simply connected, with a single ramification point of infinite order), as is the log-Riemann surface given by the Gaussian integral $(\mathbb{C}, \pi = \int e^{z^2} dz)$, which has two ramification points, both of infinite order, as in the figure below:



Log-Riemann surface of the Gaussian integral

In [BPM13], it is shown that a log-Riemann surface of finite type (which has at least one infinite order ramification point) is of the form (S', π) , where S' is a punctured compact Riemann surface $S' = S - \{p_1, \dots, p_n\}$ and π is meromorphic on S' and $d\pi$ has exponential singularities at the punctures p_1, \dots, p_n . Let h_1, \dots, h_n be the types of the exponential singularities of $d\pi$ at the punctures p_1, \dots, p_n . As described in [BPM13], each puncture p_j corresponds to an end of the log-Riemann surface where d_j infinite order ramification points are added, d_j being the order of the pole of h_j at p_j .

Let w^* be an infinite order ramification point associated to a puncture p_j . An ϵ -ball B_ϵ around w^* is isometric to the ϵ -ball around the infinite order ramification point of the Riemann surface of the logarithm (given by cutting and pasting infinitely many discs together), and there is an argument function

$$\arg_{w^*} : B_\epsilon^* \rightarrow \mathbb{R}$$

defined on the punctured ball B_ϵ^* . While the function π , which is of the form $\pi = \int e^{h_j} \alpha_j$ in a punctured neighbourhood of p_j for some meromorphic 1-form α_j , extends continuously to w^* for the metric topology on S^* , in general functions of the form $f = \int e^{h_j} \alpha$ (where α is a 1-form meromorphic near p_j) do **not** extend continuously to w^* for the metric topology ([BPM06]). Limits of these functions in sectors $\{p \in B_\epsilon^* \mid c_1 < \arg_{w^*}(p) < c_2\}$ do exist however and are independent of the sector; we say that the function is *Stolz continuous* at points of \mathcal{R} .

Definition 2.3. Define spaces of functions and 1-forms on S^* :

$$\begin{aligned}\mathcal{M}(S^*) &:= \{f \text{ meromorphic function on } S' \mid f \text{ has exponential singularities at} \\ &\quad p_1, \dots, p_n \text{ of types } h_1, \dots, h_n\} \\ \mathcal{O}(S^*) &:= \{f \in \mathcal{M}(S^*) \mid f \text{ holomorphic on } S'\} \\ \Omega(S^*) &:= \{\omega \text{ meromorphic 1-form on } S' \mid \omega \text{ has exponential singularities at} \\ &\quad p_1, \dots, p_n \text{ of types } h_1, \dots, h_n\} \\ \Omega^0(S^*) &:= \{\omega \in \Omega(S^*) \mid \omega \text{ holomorphic on } S'\}\end{aligned}$$

We remark that these are simply the spaces $\mathcal{M}_{\mathcal{H}}, \mathcal{O}_{\mathcal{H}}, \Omega_{\mathcal{H}}, \Omega_{\mathcal{H}}^0$ defined in the introduction, where $\mathcal{H} = \{[h_1], \dots, [h_n]\}$. Functions in $\mathcal{M}(S^*)$ are Stolz continuous at points of \mathcal{R} taking the value 0 there. The integrals of 1-forms ω in $\Omega_{II}(S^*)$ over curves $\gamma : [a, b] \rightarrow S^*$ joining points w_1^*, w_2^* of \mathcal{R} converge if γ is disjoint from the poles of ω and tends to these points through sectors

$$\{p \in B_\epsilon^* \mid c_1 < \arg_{w_1^*}(p) < c_2\}, \quad \{p \in B_\epsilon^* \mid c_1 < \arg_{w_2^*}(p) < c_2\}$$

(since any primitive of ω on a sector is Stolz continuous).

The definitions of the above spaces only depend on the types $\{[h_i] \in \mathcal{M}_{p_i}/\mathcal{O}_{p_i}\}$ of the exponential singularities of the 1-form $d\pi$, which do not change if $d\pi$ is multiplied by a meromorphic function. It is natural to define then a structure less rigid than that of a log-Riemann surface of finite type.

Definition 2.4 (Exp-algebraic curve). Given a punctured compact Riemann surface $S' = S - \{p_1, \dots, p_n\}$, two meromorphic functions π_1, π_2 on S' inducing log-Riemann surface structures of finite type are considered equivalent if $d\pi_1/d\pi_2$ is meromorphic on the compact surface S . An exp-algebraic curve is an equivalence class of such log-Riemann surface structures of finite type.

It follows from the uniformization theorem of [BPM13] that an exp-algebraic curve is given by the data of a punctured compact Riemann surface and n (equivalence classes of) germs of meromorphic functions $\mathcal{H} = \{[h_i] \in \mathcal{M}_{p_i}/\mathcal{O}_{p_i}\}$ with poles at the punctures.

We can associate a topological space \widehat{S} to an exp-algebraic curve, given as a set by $\widehat{S} = S' \cup \mathcal{R}$, where \mathcal{R} is the set of infinite ramification points added with respect to any map π in the equivalence class of log-Riemann surfaces of finite type, and the topology is the weakest topology such that all maps $\tilde{\pi}$ in the equivalence class extend continuously to \widehat{S} .

Finally, for a meromorphic function f on S' (respectively, meromorphic 1-form ω on S') with exponential singularities of types h_1, \dots, h_n at points p_1, \dots, p_n we can define a divisor $(f) = \sum_{p \in S} n_p \cdot p$ (respectively, $(\omega) = \sum_{p \in S} m_p \cdot p$) by $n_p = \text{ord}_p(f)$ if $p \in S'$ and $n_p = \text{ord}_{p_i}(g)$ if $p = p_i$, where g is a germ of meromorphic function at p_i such that $f = ge^{h_i}$ (respectively, $m_p = \text{ord}_p(\omega)$ if $p \in S'$ and $n_p = \text{ord}_{p_i}(\alpha)$ if $p = p_i$, where α is a germ of meromorphic 1-form at p_i such that $\omega = \alpha e^{h_i}$).

Note that the divisor (f) can also be defined by $n_p = \text{Res}(df/f, p)$, so it follows from the Residue Theorem applied to the meromorphic 1-form df/f that the divisor (f) has degree zero.

3. EXP-ALGEBRAIC CURVES AND LINE BUNDLES WITH MEROMORPHIC CONNECTIONS

Let $(S, \mathcal{H} = \{h_1, \dots, h_n\})$ be an exp-algebraic curve, where S is a compact Riemann surface of genus g and h_1, \dots, h_n are germs of meromorphic functions at points p_1, \dots, p_n . Let $\Omega(S)$ be the space of holomorphic 1-forms on S . The data \mathcal{H} defines a degree zero line bundle $L_{\mathcal{H}}$ together with a transcendental section $s_{\mathcal{H}}$ of this line bundle which is non-zero on the punctured surface S' as follows:

Solving the Mittag-Leffler problem locally for the distribution $\{h_1, \dots, h_n\}$ gives meromorphic functions on an open cover such that the differences are holomorphic on intersections, and hence gives an element of $H^1(S, \mathcal{O})$. Under the exponential this gives a degree zero line bundle as an element of $H^1(S, \mathcal{O}^*)$. Explicitly this is constructed as follows:

Let

$$B_1, \dots, B_n$$

be pairwise disjoint coordinate disks around the punctures p_1, \dots, p_n and let V be an open subset of S' intersecting each disk B_i in an annulus $U_i = V \cap B_i$ around p_i such that $\{B_1, \dots, B_n, V\}$ is an open cover of S . Define a line bundle $L_{\mathcal{H}}$ by taking the functions e^{-h_i} to be the transition functions for the line bundle on the intersections U_i . Define a holomorphic non-vanishing section of $L_{\mathcal{H}}$ on S' by:

$$s_{\mathcal{H}} := \begin{cases} 1 & \text{on } V \\ e^{-h_i} & \text{on } B_i - \{p_i\} \end{cases}$$

Define a connection $\nabla_{\mathcal{H}}$ on $L_{\mathcal{H}}$ by declaring that $\nabla_{\mathcal{H}}(s_{\mathcal{H}}) = 0$. Then for any holomorphic section s on V , $s = fs_{\mathcal{H}}$ for some holomorphic function f , and $\nabla_{\mathcal{H}}(s) = df s_{\mathcal{H}}$, so $\nabla_{\mathcal{H}}$ is holomorphic on V . On each disk B_i , letting s_i be the section which is constant equal to 1 on B_i (with respect to the trivialization on B_i), for any holomorphic section s on B_i , $s = fs_i$ for some holomorphic function f , and $s_i = e^{h_i} s_{\mathcal{H}}$, so

$$\begin{aligned} \nabla_{\mathcal{H}}(s) &= \nabla_{\mathcal{H}}(fs_i) \\ &= \nabla_{\mathcal{H}}(fe^{h_i} s_{\mathcal{H}}) \\ &= (df + f dh_i) e^{h_i} s_{\mathcal{H}} \\ &= (df + f dh_i) s_i \end{aligned}$$

thus the connection 1-form of $\nabla_{\mathcal{H}}$ with respect to s_i is given by dh_i , so $\nabla_{\mathcal{H}}$ is meromorphic on B_i with a single pole at p_i of order $d_i + 1 \geq 2$.

Let $s_{\mathcal{H}}^*$ be the unique section of the dual bundle $L_{\mathcal{H}}^*$ on S' such that $s_{\mathcal{H}} \otimes s_{\mathcal{H}}^* = 1$ on S' . Then for any non-zero meromorphic section s of $L_{\mathcal{H}}$, the function

$$f := s \otimes s_{\mathcal{H}}^*$$

is meromorphic on S' with exponential singularities at p_1, \dots, p_n of types h_1, \dots, h_n , and the divisors of s and f coincide. Thus the line bundle $L_{\mathcal{H}}$ has degree zero. In summary we have:

Theorem 3.1. *For any log-Riemann surface of finite type S^* , the line bundle $L_{\mathcal{H}}$ has degree zero and the maps*

$$s \mapsto s \otimes s_{\mathcal{H}}^*, \quad f \mapsto f \cdot s_{\mathcal{H}}$$

(respectively, $\alpha \mapsto \alpha \otimes s_{\mathcal{H}}^, \omega \mapsto \omega \cdot s_{\mathcal{H}}$) are mutually inverse isomorphisms between the spaces of meromorphic sections of $L_{\mathcal{H}}$ and $\mathcal{M}(S^*)$ (respectively, the spaces of meromorphic $L_{\mathcal{H}}$ -valued 1-forms and $\Omega(S^*)$) preserving divisors.*

In particular the vector spaces $\mathcal{M}(S^), \mathcal{O}(S^*), \Omega(S^*), \Omega_{II}(S^*), \Omega^0(S^*)$ are non-zero.*

Proof. Since the isomorphisms above preserve divisors, the spaces $\mathcal{O}(S^*), \Omega^0(S^*)$ correspond to the spaces of meromorphic sections of $L_{\mathcal{H}}$ and meromorphic $L_{\mathcal{H}}$ -valued 1-forms which are holomorphic on S' , both of which are non-empty. \square

Proposition 3.2. *The correspondence $\mathcal{H} \mapsto (L_{\mathcal{H}}, \nabla_{\mathcal{H}})$ gives a one-to-one correspondence between exp-algebraic structures on S and degree zero line bundles on S with meromorphic connections with all poles of order at least two, zero residues, and trivial monodromy.*

Proof. Since the connection 1-form of $\nabla_{\mathcal{H}}$ is given by dh_i on B_i , all residues of $\nabla_{\mathcal{H}}$ are equal to zero, while the monodromy of $\nabla_{\mathcal{H}}$ is trivial since $s_{\mathcal{H}}$ is a single-valued horizontal section.

Conversely, given such a meromorphic connection ∇ on a degree zero line bundle L , if p_1, \dots, p_n are the poles of ∇ and $\omega_1, \dots, \omega_n$ are the connection 1-forms of ∇ with respect to trivializations near p_1, \dots, p_n , then each ω_i has zero residue at p_i and pole order at least two, hence there exist meromorphic germs h_1, \dots, h_n near p_1, \dots, p_n such that $\omega_i = dh_i$. We obtain an exp-algebraic curve $(S, \mathcal{H}(L, \nabla))$.

It is clear for an exp-algebraic curve (S, \mathcal{H}) that $\mathcal{H}(L_{\mathcal{H}}, \nabla_{\mathcal{H}}) = \mathcal{H}$, so the correspondences are inverses of each other. \square

Finally we remark that by Serre Duality, the degree zero line bundle $L_{\mathcal{H}}$, given as an element of $H^1(S, \mathcal{O})$, can also be described as an element of $H^0(S, \Omega)^* = \Omega(S)^*$ using residues, as the linear functional

$$\begin{aligned} \text{Res}_{\mathcal{H}} : \Omega(S) &\rightarrow \mathbb{C} \\ \xi &\mapsto \sum_i \text{Res}(\xi \cdot h_i, p_i) \end{aligned}$$

4. TORELLI-TYPE THEOREM FOR EXP-ALGEBRAIC CURVES

We proceed to the proof of Theorem 1.1. We will need the following theorems from [Bis16] and [Gus60]:

Theorem 4.1 ([Bis16]). *Let $S^* = S' \cup \mathcal{R}$ be a log-Riemann surface of finite type, and let $H_{dR,0}^1(S^*) = \Omega^0(S^*)/d\mathcal{O}(S^*)$. Then the pairing $H_1(S^*, \mathcal{R}; \mathbb{C}) \times H_{dR,0}^1(S^*) \rightarrow \mathbb{C}$, given by integration along curves, is nondegenerate.*

Theorem 4.2 (Gusman, [Gus60]). *Let S be a compact Riemann surface and $E \subset S$ a closed subset such that $S - E$ has finitely many connected components V_1, \dots, V_m , and for each i let q_i be a point of V_i . Then any continuous function f on E which is holomorphic in the interior of E can be uniformly approximated on E by functions meromorphic on S with poles only in the set $\{q_1, \dots, q_m\}$.*

Let $(S, \mathcal{H}_1), (S, \mathcal{H}_2)$ be two exp-algebraic curves with the same underlying Riemann surface S and the same set of punctures p_1, \dots, p_n , and suppose the hypothesis of Theorem 1.1 holds, namely the line bundles $L_{\mathcal{H}_1}, L_{\mathcal{H}_2}$ are isomorphic and the induced isomorphism

$$W_{\mathcal{H}_1}^* \rightarrow W_{\mathcal{H}_2}^*$$

maps $K_{\mathcal{H}_1}$ to $K_{\mathcal{H}_2}$. Since the spaces $\mathcal{M}_{\mathcal{H}_1}, \mathcal{M}_{\mathcal{H}_2}$ are isomorphic to the spaces of meromorphic sections of $L_{\mathcal{H}_1}$ and $L_{\mathcal{H}_2}$ respectively, there is an induced isomorphism $\mathcal{M}_{\mathcal{H}_1} \rightarrow \mathcal{M}_{\mathcal{H}_2}$ which preserves divisors. We fix non-zero functions $f_i \in \mathcal{M}_{\mathcal{H}_i}, i = 1, 2$ which correspond to each other under this isomorphism, and let $S_i^* = S' \cup \mathcal{R}_i$ denote the completions induced by the corresponding log-Riemann surface structures. We also fix a meromorphic 1-form α_0 on S . Then the divisor preserving isomorphisms $\mathcal{M}_{\mathcal{H}_1} \rightarrow \mathcal{M}_{\mathcal{H}_2}$ and $\Omega_{\mathcal{H}_1} \rightarrow \Omega_{\mathcal{H}_2}$ induced by the isomorphism $L_{\mathcal{H}_1} \rightarrow L_{\mathcal{H}_2}$ can be expressed as

$$g \cdot f_1 \mapsto g \cdot f_2 \quad \text{and} \quad gf_1 \cdot \alpha_0 \mapsto gf_2 \cdot \alpha_0$$

respectively, where g varies over all meromorphic functions on S .

Lemma 4.3. *For any meromorphic function g_2 on S such that $g_2 f_2 \in \mathcal{O}_{\mathcal{H}_2}$, the 1-form $g_2 f_1 \left(\frac{df_1}{f_1} - \frac{df_2}{f_2} \right)$ lies in the space $d\mathcal{O}_{\mathcal{H}_1}$.*

Proof. The hypothesis of Theorem 1.1 implies that for any $\gamma_1 \in H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$, there is a

$$\gamma_2 \in H_1(S_2^*, \mathcal{R}_2; \mathbb{C})$$

such that $\int_{\gamma_1} gf_1 \alpha_0 = \int_{\gamma_2} gf_2 \alpha_0$ for all meromorphic functions g on S such that $gf_i \alpha_0$ is holomorphic on S' for $i = 1, 2$. If g_2 is a meromorphic function on S such that $g_2 f_2$ is holomorphic on S' , then

$$d(g_2 f_2) = gf_2 \alpha_0$$

for some meromorphic function g on S such that $gf_2 \alpha_0$ is holomorphic on S' . Since the isomorphism $\Omega(\mathcal{H}_1) \rightarrow \Omega(\mathcal{H}_2)$ is divisor preserving, we have that $gf_1 \alpha_0$ is also holomorphic on S' , so for any $\gamma_1 \in H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$ there is a $\gamma_2 \in H_1(S_2^*, \mathcal{R}_2; \mathbb{C})$ such that

$$\int_{\gamma_1} gf_1 \alpha_0 = \int_{\gamma_2} gf_2 \alpha_0 = \int_{\gamma_2} d(g_2 f_2) = 0$$

Since this is true for all $\gamma_1 \in H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$, it follows from Theorem 4.1 that $gf_1\alpha_0 \in d\mathcal{O}_{\mathcal{H}_1}$, so there exists a meromorphic function g_1 on S such that g_1f_1 is holomorphic on S' and $gf_1\alpha_0 = d(g_1f_1)$.

It follows from the equalities $gf_i\alpha_0 = d(g_if_i)$, $i = 1, 2$ that

$$dg_1 + g_1 \frac{df_1}{f_1} = g\alpha_0 = dg_2 + g_2 \frac{df_2}{f_2},$$

hence

$$dg_1 + g_1 \frac{df_1}{f_1} = \left(dg_2 + g_2 \frac{df_1}{f_1} \right) + g_2 \left(\frac{df_2}{f_2} - \frac{df_1}{f_1} \right),$$

so multiplying above by f_1 gives

$$g_2f_1 \left(\frac{df_2}{f_2} - \frac{df_1}{f_1} \right) = d(g_1f_1) - d(g_2f_1) \in d\mathcal{O}_{\mathcal{H}_1}$$

as required, since $g_1f_1, g_2f_1 \in \mathcal{O}_{\mathcal{H}_1}$ (note that $g_2f_2 \in \mathcal{O}_{\mathcal{H}_2}$ implies $g_2f_1 \in \mathcal{O}_{\mathcal{H}_1}$). \square

Proof of Theorem 1.1. We consider different cases:

Case 1. The genus of S is at least one:

In this case there exists a closed curve γ disjoint from the punctures p_1, \dots, p_n and the poles and zeroes of f_1, f_2 such that $S - \gamma$ is connected. Fix a non-zero meromorphic function g_0 on S such that g_0f_1 (and hence also g_0f_2) is holomorphic on S' .

If the meromorphic 1-form $\omega = \frac{df_1}{f_1} - \frac{df_2}{f_2}$ (which is holomorphic outside the punctures and the zeroes and poles of f_1, f_2) is not identically zero, then we can choose a continuous function u on γ such that $\int_\gamma u g_0 f_1 \omega \neq 0$ (since the 1-form $g_0 f_1 \omega$ is holomorphic and not identically zero on γ). By Theorem 4.2, since $S - \gamma$ is connected and contains p_1 , we can choose a meromorphic function v on S which is holomorphic on $S - \{p_1\}$ and uniformly close enough to u on γ such that $\int_\gamma v g_0 f_1 \omega \neq 0$. Letting $g_2 = v g_0$, we have that $g_2 f_2$ is holomorphic on S' and $\int_\gamma g_2 f_1 \omega \neq 0$, contradicting Lemma 4.3. It follows that $df_1/f_1 \equiv df_2/f_2$, from which it follows easily that $\mathcal{H}_1 = \mathcal{H}_2$.

Case 2. The genus of S is zero and the number n of punctures is at least two:

In this case $S = \widehat{\mathbb{C}}$ and we may assume $p_1 = 0, p_2 = \infty$. Fix a non-zero polynomial P such that Pf_1, Pf_2 are holomorphic on S' . Then by Lemma 4.3, for all $k \in \mathbb{Z}$, taking $g_2 = z^k P(z)$ we have

$$\text{Res}((z^k P) f_1 \omega, 0) = 0$$

from which it follows that the Laurent series of $Pf_1\omega$ around $z = 0$ vanishes identically, hence $\omega \equiv 0$ and $\mathcal{H}_1 = \mathcal{H}_2$.

Case 3. The genus of S is zero and there is only one puncture:

In this case $S = \widehat{\mathbb{C}}$ and we may assume the single puncture $p_1 = \infty$, and that the functions f_1, f_2 are of the form $f_i = e^{P_i}$ for some polynomials P_1, P_2 . In this case it follows from the main theorem of [Bis16] that the dimension of $H_1(S_i^*, \mathcal{R}_i; \mathbb{C})$ equals $\deg(P_i) - 1$. Since $K_{\mathcal{H}_1}$ and $K_{\mathcal{H}_2}$ are isomorphic by hypothesis, it follows that $\deg(P_1) = \deg(P_2) = d$ say, where $d \geq 2$ since $H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$ is non-trivial.

Let $P_1(z) = a_d z^d + \dots + a_0$, $P_2(z) = b_d z^d + \dots + b_0$. Let $\gamma_1, \dots, \gamma_{d-1}$ be a basis for $H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$ as described in section 4 of [Bis16], each γ_k being a curve joining a pair of ramification points w_0^*, w_k^* , where $\mathcal{R}_1 = \{w_0^*, \dots, w_{d-1}^*\}$. By hypothesis, for each curve $\gamma_k \in H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$ there is a $\gamma'_k \in H_1(S_2^*, \mathcal{R}_2; \mathbb{C})$ such that

$$\int_{\gamma_k} Q(z) e^{P_1(z)} dz = \int_{\gamma'_k} Q(z) e^{P_2(z)} dz$$

for all polynomials Q . Consider the $(d-1) \times d$ matrix

$$\begin{aligned} M &= \begin{pmatrix} \int_{\gamma_1} e^{P_1(z)} dz & \dots & \int_{\gamma_1} z^{d-2} e^{P_1(z)} dz & \int_{\gamma_1} z^{d-1} e^{P_1(z)} dz \\ \vdots & \ddots & \vdots & \vdots \\ \int_{\gamma_{d-1}} e^{P_1(z)} dz & \dots & \int_{\gamma_{d-1}} z^{d-2} e^{P_1(z)} dz & \int_{\gamma_{d-1}} z^{d-1} e^{P_1(z)} dz \end{pmatrix} \\ &= \begin{pmatrix} \int_{\gamma'_1} e^{P_2(z)} dz & \dots & \int_{\gamma'_1} z^{d-2} e^{P_2(z)} dz & \int_{\gamma'_1} z^{d-1} e^{P_2(z)} dz \\ \vdots & \ddots & \vdots & \vdots \\ \int_{\gamma'_{d-1}} e^{P_2(z)} dz & \dots & \int_{\gamma'_{d-1}} z^{d-2} e^{P_2(z)} dz & \int_{\gamma'_{d-1}} z^{d-1} e^{P_2(z)} dz \end{pmatrix} \end{aligned}$$

It follows from Theorem III.1.5.1 of [BPM06] that the $(d-1)$ 1-forms

$$z^k e^{P_1} dz, \quad k = 0, \dots, (d-2)$$

span $H_{dR,0}^1(S_1^*)$, and hence form a basis for $H_{dR,0}^1(S_1^*)$. Since by Theorem 4.1 the pairing with $H_1(S_1^*, \mathcal{R}_1; \mathbb{C})$ is nondegenerate, it follows that the $(d-1) \times (d-1)$ submatrix formed by the first $(d-1)$ columns of M is nonsingular, thus M has rank $(d-1)$. On the other hand, since $d(e^{P_i}) = P'_i e^{P_i} dz$, $i = 1, 2$, it follows that

$$M \cdot \begin{pmatrix} a_1 \\ \vdots \\ (d-1)a_{d-1} \\ da_d \end{pmatrix} = M \cdot \begin{pmatrix} b_1 \\ \vdots \\ (d-1)b_{d-1} \\ db_d \end{pmatrix} = 0$$

hence there is a scalar λ such that $kb_k = \lambda ka_k$, $k = 1, \dots, d-1$, so $P'_2 = \lambda P'_1$. It follows from Lemma 4.3 that for any polynomial Q the 1-form

$$Q e^{P_1} (\lambda - 1) P'_1 dz = Q e^{P_1} (P'_2 - P'_1) dz$$

lies in $d\mathcal{O}_{\mathcal{H}_1}$. Thus if $\lambda \neq 1$, then $Q P'_1 e^{P_1} dz \in d\mathcal{O}_{\mathcal{H}_1}$, hence

$$Q' e^{P_1} dz = d(Q e^{P_1}) - Q P'_1 e^{P_1} dz \in d\mathcal{O}_{\mathcal{H}_1}$$

for all polynomials Q . Since all 1-forms in $\Omega_{\mathcal{H}_1}^0$ are of the form $P e^{P_1} dz$ for some polynomial P and any $P = Q'$ for some polynomial Q , it follows that $H_{dR,0}^1(S_1^*)$ is trivial, a contradiction. Thus $\lambda = 1$, so $P'_2 = P'_1$ and hence $\mathcal{H}_1 = \mathcal{H}_2$. \square

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